Journal of Geometry and Physics 10 (1993) 381–392
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Cohomological obstructions to the equivariant extension of closed invariant forms

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Received 18 August 1992

We study under what condition a closed invariant form on a manifold with a group action admits an equivariant extension. We derive a sequence of obstructions in the cohomology groups of the Lie algebra with coefficients in appropriate modules. We illustrate the result with two specific examples. We then discuss when such cohomological obstructions vanish. Finally, we compare our analysis with the spectral sequence point of view.

Keywords: *Equivariant cohomology, extensions, cohomology of Lie algebras, spectral sequences*
1991 MSC: *55N91, 17B56, 18G15, 55R20.*

1. Introduction

The theory of equivariant cohomology has several important and well-known applications in symplectic geometry. For example, it explains, almost from the definition, the variation of symplectic structures of the quotients from a hamiltonian torus action [3]. Equivariant extensions of the symplectic form were found to be in 1-1 correspondence with the Poisson liftings of the hamiltonian action [1]. And the "localization theorem" provides a penetrating understanding [2,1] of the Duistermaat-Heckman formula [3]. Recently the theory comes into contact with other areas of mathematical physics, shedding new light on problems such as the gauging of WZW model [9] (and possibly many other physical models), BRST quantization [6], and "non-abelian localizaion" [10], which may lead to a non-abelian exact stationary phase formula.

The purpose of this paper is to address a related yet rather basic question, namely, under what condition a closed invariant form admits an equivariant extension. We first look at two examples from different backgrounds mentioned above. Throughout this paper, we denote $\mathfrak{g}$ the Lie algebra of a finite dimensional connected real Lie group G. We choose a basis $\{x_a, a = 1, \dots, \dim \mathfrak{g}\}$ of $\mathfrak{g}$ and fix the structure constants according to $[x_a, x_b] = c^c_{ab} x_c$. (We sum over repeated indices unless otherwise stated.) If G acts smoothly on a finite dimen-

0393-0440/93/$ 06.00

sional manifold M, we write i_a and L_a respectively for the contraction and the Lie derivative along the vector field induced by x_a.

Example 1. Suppose (M, ω) is a symplectic manifold with a hamiltonian G-action. G invariance $L_a\omega = 0$ and closedness $\mathrm{d}\omega = 0$ imply that $\mathrm{d}i_a\omega = 0$, i.e., $i_a\omega$ is a closed one-form. If they are exact, i.e., $i_a\omega = \mathrm{d}f_a$, the functions f_a (known as the moment maps) do not necessarily transform according to the adjoint representation of $\mathfrak{g}$. In other words the linear map $x_a \mapsto f_a$ from $\mathfrak{g}$ to $C^\infty(M)$ with the Poisson bracket $\{f_a, f_b\} = L_a f_b$ need not be a homomorphism of Lie algebras. Nevertheless, the differences

$$\gamma_{ab} = \{f_a, f_b\} - c_{ab}^c f_c \tag{1.1}$$

are a set of constant functions (assuming that M is connected) and satisfy

$$c_{ab}^d \gamma_{dc} + c_{bc}^d \gamma_{da} + c_{ca}^d \gamma_{db} = 0. \tag{1.2}$$

If

$$\gamma_{ab} = c_{ab}^c \beta_c \tag{1.3}$$

for a set of constants β_a, which is clearly a solution of (1.2) due to the Jacobi identity, then we can redefine the moment maps by $f_a' = f_a - \beta_a$ so that $i_a\omega = \mathrm{d}f_a'$ and $L_a f_b' = \{f_a', f_b'\} = c_{ab}^c f_c'$, i.e., the map $x_a \mapsto f_a'$ *is* a Lie algebra homomorphism. It is well-known that (1.2) and (1.3) are the cocycle and coboundary conditions for the cohomology group $H^2(\mathfrak{g}, \mathbb{R})$ of the Lie algebra.

Example 2. This appears in the understanding of the Lagrangian action of the gauged Wess-Zumino–Witten (WZW) conformal field theory (see ref. [9] and references therein). Let ω be a closed invariant three-form on a manifold M with a G-action. If B is a three-manifold, then for each map $\varphi: B \to M$, $\varphi^*\omega$ is a top form on B. (In the WZW model, $M = G$ with the left and right multiplications by G and

$$\omega = \frac{1}{12\pi} \mathrm{Tr}\,(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g), \tag{1.4}$$

where $g^{-1}\mathrm{d}g$ is the Maurer–Cartan form of G. $\int_B \varphi^*\omega$ is known as the Wess-Zumino term.) To *gauge* the theory means, at least mathematically, to find a natural way of defining $\varphi^*\omega$ if φ is now a global section of a (possibly trivial) fiber bundle over B with structure group G and fiber M. This is possible if and only if [9] the closed forms $i_a\omega$ are exact, i.e., $i_a\omega = \mathrm{d}\lambda_a$, and the λ_a's transform according to the adjoint representation of $\mathfrak{g}$, i.e.,

$$L_a\lambda_b = c_{ab}^c \lambda_c, \tag{1.5}$$

and they satisfy

$$i_a\lambda_b + i_b\lambda_a = 0. \tag{1.6}$$

One can show that $\gamma_{ab} = L_a\lambda_b - c^c_{ab}\lambda_c$ is a closed one-form satisfying a similar "cocycle" condition

$$c^d_{ab}\gamma_{dc} - c^d_{bc}\gamma_{ad} + c^d_{ac}\gamma_{bd} = L_a\gamma_{bc} - L_b\gamma_{ac}. \tag{1.7}$$

If γ is a "coboundary" in the sense

$$\gamma_{ab} = L_a\beta_b - c^c_{ab}\beta_c \tag{1.8}$$

for a set of *closed* one-forms β_a, then we can redefine $\lambda'_a = \lambda_a - \beta_a$ so that they satisfy $i_a\omega = \mathrm{d}\lambda'_a$ and (1.5): $L_a\lambda'_b = c^c_{ab}\lambda'_c$.

As pointed out in refs. [1,9], examples 1 and 2 are about equivariant extensions of closed two- and three-forms respectively. In the first case the obstruction to such an extension lies in $H^2(\mathfrak{g},\mathbb{R})$; this is well-known. In example 2 the appropriate spaces on which (1.7) and (1.8) are cocycle and coboundary conditions are yet to be identified; we will show that the obstruction lies in a suitable Lie algebra cohomology group.

In section 2, we recall the basic definitions in equivariant cohomology theory and its de Rham versions. Section 3 is devoted to studying the conditions for the existence of equivariant extensions. We show that there is a sequence of obstructions in the cohomology groups of the Lie algebra $\mathfrak{g}$ with coefficients in suitable $\mathfrak{g}$-modules. Our general formula specializing to example 2 yields (1.7) and (1.8). In section 4, we discuss when the cohomological obstructions vanish. We then show that our result agrees with the spectral sequence point of view.

2. De Rham models for equivariant cohomology

In this section, we introduce the basic setting and notations that we need to formulate the problem of equivariant extensions.

Let G be a Lie group and $G \hookrightarrow EG \to BG$ the universal G-bundle. The total space EG is contractible and has a right G-action; the quotient BG is the classifying space of G-bundles. If G acts smoothly on a manifold M, the *Borel construction* on M is the space $M_G = EG \times_G M$, where the identification is $(e,m) \sim (eg, gm)$ for $e \in EG$, $m \in M$, $g \in G$. The projection $M_G \to BG$ is a fibration over BG with fiber M; $M_G \to M/G$ is not a smooth fibration in general unless G acts freely on M, in which case each fiber is EG. The qth *equivariant cohomology group* of M (with the G-action) is $H^q_G(M) = H^q(EG \times_G M)$. It can be non-zero for arbitrarily large q.

Let $\mathfrak{g}$ be the Lie algebra of G with a basis $\{x_a\}$. The *Weil algebra* of $\mathfrak{g}$ is the set $W(\mathfrak{g}) = \bigwedge(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*)$ with a grading specified by $\deg \bigwedge^1(\mathfrak{g}^*) = 1$, $\deg S^1(\mathfrak{g}^*) = 2$. Let $\{x^{*a}\}$ be the dual basis of $\mathfrak{g}^*$ and θ^a, ϕ^a, the images of x^{*a} in $\bigwedge^1(\mathfrak{g}^*)$, $S^1(\mathfrak{g}^*)$ respectively. We define the formal contraction i_a with x_a and the exterior derivative d by

$$i_a\theta^b = \delta^b_a, \quad i_a\phi^b = 0, \tag{2.1}$$

$$\mathrm{d}\theta^a = \phi^a - \tfrac{1}{2}c^a_{bc}\theta^b\theta^c, \quad \mathrm{d}\phi^a = c^a_{bc}\phi^b\theta^c. \tag{2.2}$$

One can check that d and i_a satisfy the standard relations and the Lie derivative $L_a = \mathrm{d}i_a + i_a\mathrm{d}$ on $W(\mathfrak{g})$ is identical to the coadjoint action. If $P \xrightarrow{\pi} B$ is a principal G-bundle with a connection, then the *Weil homomorphism* w: $W(\mathfrak{g}) \to \Omega(P)$ of the two graded algebras maps θ^a and ϕ^a to the corresponding components of connection and curvature respectively. w commutes with d, i_a, L_a of the two algebras. We define the *basic* subspace of $\Omega(P)$ by

$$\Omega(P)_{\mathrm{bas}} = \{\omega \in \Omega(P) \,|\, i_x\omega = 0, L_x\omega = 0, \forall x \in \mathfrak{g}\}. \tag{2.3}$$

It is clear that d maps $\Omega(P)_{\mathrm{bas}}$ into itself and that $\Omega(B) \xrightarrow{\pi^*} \Omega(P)_{\mathrm{bas}}$ is an isomorphism. One can define $W(\mathfrak{g})_{\mathrm{bas}}$ similarly and show that it is equal to $S(\mathfrak{g}^*)^G$.

If M is a smooth manifold with a G-action, the space of *equivariant forms* on M is

$$\Omega_{\mathfrak{g}}(M) = (\Omega(M) \otimes W(\mathfrak{g}))_{\mathrm{bas}}. \tag{2.4}$$

The *Chern–Weil homomorphism*

$$\Omega_{\mathfrak{g}}(M) = (\Omega(M) \otimes W(\mathfrak{g}))_{\mathrm{bas}} \to \Omega(P \times M)_{\mathrm{bas}} \cong \Omega(P \times_G M). \tag{2.5}$$

is the restriction to the basic subspace of the map from $\Omega(M) \otimes W(\mathfrak{g})$ to $\Omega(P \times M)$, a trivial extension of the Weil homomorphism. The induced homomorphism on the cohomology group $\overline{w}$: $H^*(\Omega_{\mathfrak{g}}(M), \mathrm{d}) \to H^*(P \times_G M)$ does not depend on the choice of connection on P. Since $EG \to BG$ can be approximated by finite dimensional fibrations, there is a natural homomorphism $H^*(\Omega_{\mathfrak{g}}(M), \mathrm{d}) \to H^*(EG \times_G M) = H^*_G(M)$. It turns out that if G is a compact connected Lie group, this homomorphism is an isomorphism [1]. Thus $(\Omega_{\mathfrak{g}}(M), \mathrm{d})$ is a "de Rham model" for equivariant cohomology.

Another model, due to Cartan, is probably more useful for computations. Consider the isomorphism [7] (see also ref. [1] for the case $G = U(1)$)

$$\Omega_{\mathfrak{g}}(M) \underset{j}{\overset{\epsilon}{\rightleftarrows}} (\Omega(M) \otimes S(\mathfrak{g}^*))^G, \tag{2.6}$$

where the maps are (for $a = 1, \ldots, \dim\mathfrak{g}$ and $\omega \in \Omega(M)$)

$$\epsilon(\theta^a) = 0, \quad \epsilon(\phi^a) = \phi^a, \quad \epsilon(\omega) = \omega \tag{2.7}$$

and

$$j(\phi^a) = \phi^a, \quad j(\omega) = \omega - \theta^a i_a\omega + \frac{1}{2!}\theta^a\theta^b i_a i_b\omega - \cdots. \tag{2.8}$$

The induced differential $\tilde{\mathrm{d}} = \epsilon\,\mathrm{d}j$ on $(\Omega(M) \otimes S(\mathfrak{g}^*))^G$ is

$$\tilde{\mathrm{d}}\phi^a = 0, \quad \tilde{\mathrm{d}}\omega = \mathrm{d}\omega - \phi^a i_a\omega. \tag{2.9}$$

Hence, we have

$$H^*(\Omega_{\mathfrak{g}}(M), \mathrm{d}) \cong H^*((\Omega(M) \otimes S(\mathfrak{g}^*))^G, \tilde{\mathrm{d}}); \tag{2.10}$$

both of them are isomorphic to $H_G^*(M)$ if G is compact.

It is now straightforward to explain the extension problem. If $\omega \in \Omega^q(M)$ is a closed G-invariant q-form on M with a G-action, a *(closed) equivariant extension* of ω is an element $\omega^\# \in \Omega_{\mathfrak{g}}^q(M)$ such that

$$\mathrm{d}\omega^\# = 0 \quad \text{and} \quad \omega^\#|_{\theta=0,\phi=0} = \omega; \tag{2.11}$$

or equivalently, an element $\tilde{\omega} \in (\Omega(M) \otimes S(\mathfrak{g}^*))^G$ of degree q such that

$$\tilde{\mathrm{d}}\tilde{\omega} = 0 \quad \text{and} \quad \tilde{\omega}|_{\phi=0} = \omega. \tag{2.12}$$

Geometrically, if such an extension $\omega^\#$ or $\tilde{\omega}$ exists, then for any principal G-bundle $P \xrightarrow{\pi} B$, there is a closed q-form $\overline{\omega}$ on the total space of $P \times_G M \to B$ whose restriction to each fiber is ω. Therefore, if $\varphi: B \to P$ is a global section, the pull back $\varphi^*\overline{\omega}$ would be the closest analogue of $\varphi^*\omega$ when $\varphi : B \to M$ was a map. This is needed in gauging the Wess-Zumino term. Notice that neither the definition nor the geometric consequence of the equivariant extension requires the compactness of G.

3. Cohomological obstructions to equivariant extension

Assume that a closed G-invariant q-form $\omega \in \Omega^q(M)$ has an equivariant extension, that is, there exists a closed form

$$\tilde{\omega} = \sum_{r=0}^{[q/2]} \omega^{(q-2r)}_{a_1\cdots a_r} \phi^{a_1} \cdots \phi^{a_r} \tag{3.1}$$

in $(\Omega^q(M) \otimes S(\mathfrak{g}^*))^G$ such that $\omega^{(q)} = \omega$. $\tilde{\mathrm{d}}\tilde{\omega} = 0$ implies a series of identities

$$i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}} = \mathrm{d}\omega^{(q-2r)}_{a_1\cdots a_r} \quad (r = 1, 2, \cdots, [q/2] + 1), \tag{3.2}$$

or

$$\begin{aligned}
0 &= \mathrm{d}\omega^{(q)}, \\
i_{a_1}\omega^{(q)} &= \mathrm{d}\omega^{(q-2)}_{a_1}, \\
i_{\{a_1}\omega^{(q-2)}_{a_2\}} &= \mathrm{d}\omega^{(q-4)}_{a_1 a_2}, \\
&\cdots\cdots \\
i_{\{a_1}\omega^{(q-2[q/2]+2)}_{a_2\cdots a_{[q/2]}\}} &= \mathrm{d}\omega^{(q-2[q/2])}_{a_1\cdots a_{[q/2]}}, \\
i_{\{a_1}\omega^{(q-2[q/2])}_{a_2\cdots a_{[q/2]+1}\}} &= 0, \quad \text{if } q \text{ is odd.}
\end{aligned} \tag{3.3}$$

(The brace stands for symmetrization of tensor indices.) First of all, (3.3) implies

Condition (A_r). *Each* $(q-2r+1)$*-form* $i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}}$ *is exact.*

Furthermore, since $\tilde{\omega}$ is G-invariant, the $(q-2r)$-forms $\omega^{(q-2r)}_{a_1\cdots a_r}$ must transform according to the rth totally symmetrized tensor product $S^r\mathfrak{g}$ of the adjoint representation of $\mathfrak{g}$. Therefore, (3.3) also implies

Condition (B_r). *Each* $\mathfrak{g}$*-map* $S^r\mathfrak{g} \to B^{q-2r+1}(M)$ *defined by the tensor* $i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}}$ *lifts to a* $\mathfrak{g}$*-map* $S^r\mathfrak{g} \to \Omega^{q-2r}(M)$ *defining a new tensor* $\omega^{(q-2r)}_{a_1\cdots a_r}$.

(A_r) and (B_r), being trivially true when $r=0$, can be regarded as a set of recursion conditions as r runs through $1,\cdots,[q/2]+1$. It is clear that they are also sufficient for the existence of $\omega^{(q-2r)}$ and hence $\tilde{\omega}$.

We now analyse the two conditions. Assuming (A_{r-1}) and (B_{r-1}) are already satisfied, it is automatic that each $(q-2r+1)$-form $i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}}$ is closed. In fact, from

$$L_{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r} = \sum_{i=1}^{r} c^{a'}_{a_1a_i}\omega^{(q-2r+2)}_{a_2\cdots a_{i-1}a'a_{i+1}\cdots a_r}, \tag{3.4}$$

we have,

$$L_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}} = 0, \tag{3.5}$$

since the structure constants are anti-symmetric in the two lower indices. It follows that

$$\mathrm{d}i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}} = -i_{\{a_1}\mathrm{d}\omega^{(q-2r+2)}_{a_2\cdots a_r\}} = -i_{\{a_1}i_{a_2}\omega^{(q-2r+4)}_{a_3\cdots a_r\}} = 0, \tag{3.6}$$

because any two contractions anti-commute. Thus condition (A_r) requires that the de Rham class in $H^{q-2r+1}(M)$ represented by the closed form $i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}}$ is trivial.

Condition (B_r) needs more explanation. Since d commutes with the G-action on differential forms,

$$0 \to Z^{q-2r}(M) \to \Omega^{q-2r}(M) \xrightarrow{\mathrm{d}} B^{q-2r+1}(M) \to 0 \tag{3.7}$$

is a short exact sequence of $\mathfrak{g}$-modules (i.e., modules over the universal enveloping algebra $U(\mathfrak{g})$). Condition (B_r) requires that the element $i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}}$ in $\mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, B^{q-2r+1}(M))$ should come from one in $\mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, \omega^{q-2r}(M))$. Using the long exact sequence

$$\begin{aligned} 0 \to\ & \mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, Z^{q-2r}(M)) \to \mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, \Omega^{q-2r}(M)) \to \\ & \mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, B^{q-2r+1}(M)) \xrightarrow{\partial} \mathrm{Ext}^1_{U(\mathfrak{g})}(S^r\mathfrak{g}, Z^{q-2r}(M)) \to \cdots, \end{aligned} \tag{3.8}$$

we conclude that (B_r) is satisfied if and only if the image of $i_{\{a_1}\omega^{(q-2r+2)}_{a_2\cdots a_r\}}$ in $\mathrm{Ext}^1_{U(\mathfrak{g})}(S^r\mathfrak{g}, Z^{q-2r}(M))$ is zero.

It is a standard result (see for example [5, §3]) that there is a natural isomorphism

$$\mathrm{Ext}^n_{U(\mathfrak{g})}(W, V) \cong H^n(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(W, V)), \quad n = 0, 1, 2, \cdots, \tag{3.9}$$

for two $\mathfrak{g}$-modules W and V. Since

$$\mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^n\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(W, V)) \cong \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^n\mathfrak{g} \otimes_{\mathbb{R}} W, V), \tag{3.10}$$

$H^n(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(W, V))$ is then the cohomology of the complex

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^0\mathfrak{g} \otimes_{\mathbb{R}} W, V) &\xrightarrow{\delta_0} \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^1\mathfrak{g} \otimes_{\mathbb{R}} W, V) \\ &\xrightarrow{\delta_1} \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^2\mathfrak{g} \otimes_{\mathbb{R}} W, V) \to \cdots, \end{aligned} \tag{3.11}$$

where $\delta_n: \mathrm{Hom}_{\mathbb{R}}(\bigwedge^n\mathfrak{g} \otimes_{\mathbb{R}} W, V) \to \mathrm{Hom}_{\mathbb{R}}(\bigwedge^{n+1}\mathfrak{g} \otimes_{\mathbb{R}} W, V)$ is the coboundary map. For $\gamma_n \in \mathrm{Hom}_{\mathbb{R}}(\bigwedge^n\mathfrak{g} \otimes_{\mathbb{R}} W, V)$, $w \in W$ and $x_i \in \mathfrak{g}$ $(i = 0, \cdots, n)$,

$$\begin{aligned} &\delta_n\gamma_n(x_0 \wedge \cdots \wedge x_n \otimes w) \\ &= \sum_{i=0}^{n} (-1)^i (x_i\gamma_n(x_0 \wedge \cdots \widehat{x}_i \cdots \wedge x_n \otimes w) - \gamma_n(x_i x_0 \wedge \cdots \widehat{x}_i \cdots \wedge x_n \otimes w)) \\ &\quad + \sum_{0 \le i < j \le n} (-1)^{i+j} \gamma_n([x_i, x_j] \wedge x_0 \wedge \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots \wedge x_n \otimes w). \end{aligned} \tag{3.12}$$

To describe the natural isomorphism in (3.9), recall that "Ext^n" is the nth derived functor of "Hom". Consider the projective resolution of W (compare [5, §2])

$$0 \leftarrow W \xleftarrow{\epsilon} U_0 \otimes_{\mathbb{R}} W \xleftarrow{\partial_0} U_1 \otimes_{\mathbb{R}} W \xleftarrow{\partial_1} U_2 \otimes_{\mathbb{R}} W \leftarrow \cdots. \tag{3.13}$$

Here $U_n = U(\mathfrak{g}) \otimes_{\mathbb{R}} \bigwedge^n\mathfrak{g}$; as a $\mathfrak{g}$-module, $\mathfrak{g}$ acts on $U(\mathfrak{g})$ as usual and acts trivially on $\bigwedge^n\mathfrak{g}$. The map ϵ picks up the constant term of degree zero in $U(\mathfrak{g})$ and ∂_n is defined by

$$\begin{aligned} \partial_n(u \otimes x_0 \wedge \cdots \wedge x_n) &= \sum_{i=1}^{n} (-1)^i u x_i \otimes x_0 \wedge \cdots \widehat{x}_i \cdots \wedge x_n \\ &\quad + \sum_{0 \le i < j \le n} u \otimes [x_i, x_j] \wedge x_0 \wedge \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots \wedge x_n \\ &\quad (u \in U(\mathfrak{g}),\ x_i \in \mathfrak{g},\ i = 0, \cdots, n) \end{aligned} \tag{3.14}$$

on U_{n+1} and extends trivially to $U_{n+1} \otimes W$. $\mathrm{Ext}^n_{U(\mathfrak{g})}(W, V)$ is defined as the nth cohomology group of the cochain complex

$$\begin{aligned} 0 \to \mathrm{Hom}_{U(\mathfrak{g})}(U_0 \otimes_{\mathbb{R}} W, V) &\xrightarrow{\partial_0^*} \mathrm{Hom}_{U(\mathfrak{g})}(U_1 \otimes_{\mathbb{R}} W, V) \\ &\xrightarrow{\partial_1^*} \mathrm{Hom}_{U(\mathfrak{g})}(U_2 \otimes_{\mathbb{R}} W, V) \to \cdots \end{aligned} \tag{3.15}$$

obtained by applying the functor $\mathrm{Hom}_{U(\mathfrak{g})}(\cdot, V)$ to (3.13). Under the natural isomorphism

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g})}(U_n \otimes_{\mathbb{R}} W, V) &= \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{\mathbb{R}} \textstyle\bigwedge^n \mathfrak{g} \otimes_{\mathbb{R}} W, V) \\ &\cong \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}), \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^n \mathfrak{g} \otimes_{\mathbb{R}} W, V)) \cong \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^n \mathfrak{g} \otimes_{\mathbb{R}} W, V),\end{aligned} \tag{3.16}$$

the map ∂_n^* corresponds to δ_n. Hence the isomorphism in (3.9).

We summarize our discussions in the following

Theorem 1. *If a Lie group G acts smoothly on a manifold M and ω is a closed G-invariant q-form on M, then ω admits an equivariant extension if and only if the following sequence of conditions are satisfied recursively for $r = 1, \cdots, [(q+1)/2]$ (set $\omega^{(q)} = \omega$):*
(A_r) The de Rham class in $H^{q-2r+1}(M)$ represented by the closed form $i_{\{a_1}\omega^{(q-2r+2)}_{a_2 \cdots a_r\}}$ is trivial;
(B_r) The cohomological class in $H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(S^r\mathfrak{g}, Z^{q-2r}(M)))$ defined by the image of $i_{\{a_1}\omega^{(q-2r+2)}_{a_2 \cdots a_r\}}$ under the map $\mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, B^{q-2r+1}(M)) \xrightarrow{\partial} \mathrm{Ext}^1_{U(\mathfrak{g})}(S^r\mathfrak{g}, Z^{q-2r}(M)) \xrightarrow{\cong} H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(S^r\mathfrak{g}, Z^{q-2r}(M)))$ is trivial, thereby defining a new tensor $\omega^{(q-2r)}_{a_1 \cdots a_r}$.

In example 2, $\omega^{(3)} = \omega$, $\omega^{(1)}_a = \lambda_a$ and the last identity of (3.3) coincide with (1.6). We shall check from the cochain complex

$$\begin{aligned}\cdots \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, Z^1(M)) &\xrightarrow{\delta_0} \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g} \otimes \mathfrak{g}, Z^1(M)) \\ &\xrightarrow{\delta_1} \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^2 \mathfrak{g} \otimes \mathfrak{g}, Z^1(M)) \to \cdots\end{aligned} \tag{3.17}$$

the cocycle and coboundary conditions for the cohomology group $H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(S^1\mathfrak{g}, Z^1(M))) = \ker\delta_1/\mathrm{im}\delta_0$. Using (3.12), $\{\gamma_{ab}\} \in \ker\delta_1$ means

$$0 = (\delta_1 \gamma)_{abc} = L_a \gamma_{bc} - L_b \gamma_{ac} - c^d_{ac}\gamma_{bd} + c^d_{bc}\gamma_{ad} - c^d_{ab}\gamma_{dc}; \tag{3.18}$$

whereas $\{\gamma_{ab}\} \in \mathrm{im}\delta_0$ means

$$\gamma_{ab} = (\delta_0 \beta)_{ab} = L_a \beta_b - c^c_{ab}\beta_c, \tag{3.19}$$

for $\beta_a \in Z^1(M)$. These identities match precisely (1.7) and (1.8).

For example 1, we only have to replace the set $Z^1(M)$ in (3.17) by $Z^0(M) = \mathbb{R}$. As a consequence, the terms in (3.18) and (3.19) involving Lie derivatives drop out; the remaining terms are totally anti-symmetric in the tensor indices, since $\gamma_{ab} = -\gamma_{ba}$ as defined in (1.1). Hence (3.18) and (3.19) are reduced to (1.2) and (1.3) respectively. Furthermore, δ_1, δ_0 are equal (up to a sign) to δ_3, δ_2 of a different cochain complex

$$\cdots \to \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R}) \xrightarrow{\delta_2} \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^2 \mathfrak{g}, \mathbb{R}) \xrightarrow{\delta_3} \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^3 \mathfrak{g}, \mathbb{R}) \to \cdots \tag{3.20}$$

whose cohomology group is $H^2(\mathfrak{g}, \mathbb{R}) = \ker\delta_3/\mathrm{im}\delta_2$. Thus we get a pleasant surprise:

$$H^2(\mathfrak{g}, \mathbb{R}) \cong \mathrm{Ext}^1_{U(\mathfrak{g})}(\mathfrak{g}, \mathbb{R}). \tag{3.21}$$

This coincidence may be explained by the following observation. $\mathrm{Ext}^1_{U(\mathfrak{g})}(W, V)$ is the set of inequivalent $\mathfrak{g}$-module extensions of W by V; whereas $H^2(\mathfrak{g}, \mathfrak{a})$ is the set of Lie algebra extensions of $\mathfrak{g}$ by another Lie algebra $\mathfrak{a}$. If $W = \mathfrak{g}$ and $V = \mathfrak{a}$, a trivial $\mathfrak{g}$-module (or abelian Lie algebra), the two notions of extensions are the same. Hence the isomorphism in (3.21).

The problem of uniqueness is relatively easy. If the equivariant extension of ω exists and $\omega^{(q)}, \cdots, \omega^{(q-2r+2)}$ are already chosen, then the set of different choices of $\omega^{(r)}$ is parametrized by $\mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, Z^{q-2r}(M))$. For instance, the f_a's in example 1 and λ_a's in example 2 are unique up to a set of constants and closed one-forms respectively in the adjoint representation. The cohomology class of $\tilde{\omega}$ in $H^q_G(M)$ does depend on the choices of the $\omega^{(q-2r)}$'s.

4. Further discussions

In this section, we first discuss when the cohomological obstructions found in theorem 1 vanish.

The conditions (A_r) are about topologies of the manifold M; they need not be true even if G is compact and semi-simple. The counterpart in symplectic geometry is that the hamiltonian actions need not lift to moment maps; such examples are well-known. Though $H^{q-2r+1}(M) = 0$ is certainly a sufficient condition for (A_r), this is impossible if $q-2r+1 = 0$, which happens in gauging the WZW model. The three-form (1.4) on $M = G$ is invariant under the group $G \times G$ acting as the left and right multiplications. Trivially, it is invariant under any group H provided there is a group homomorphism from H to $G \times G$. It was shown that the $G \times G$-equivariant extension does not exist [9], because (A_2) fails to hold. Nevertheless if $H = G$ or its subgroup with the diagonal imbedding into $G \times G$, then the H-equivariant extension of ω does exist [9].

On the other hand, the conditions (B_r) are about properties of the group G. If G is semi-simple and E, F are finite dimensional representations of its Lie algebra $\mathfrak{g}$, then $\mathrm{Ext}^1_{U(\mathfrak{g})}(E, F) = 0$ [5, §11]. It follows that if G is compact, semi-simple, the space $\mathrm{Ext}^1_{U(\mathfrak{g})}(S^r\mathfrak{g}, Z^{q-2r}(M))$ is always zero, since the representation on $Z^{q-2r}(M)$ is a direct sum of finite dimensional ones. If G is semi-simple but non-compact, there are a few "vanishing theorems" for the group "Ext^n" [5, §11], but applications to our case are not straightforward, because $Z^{q-2r}(M)$ may be fairly complicated as a representation space. A notable exception (from symplectic geometry) is that semi-simplicity implies $\mathrm{Ext}^1_{U(\mathfrak{g})}(\mathfrak{g}, Z^0(M)) = 0$, but it probably does not guarantee (B_r) if $q - 2r > 0$. If G is a general Lie group, the following is perhaps the simplest example beyond symplectic geometry in which the equivariant extension does not exist while all the conditions (A_r) are true. Take $G = \mathbb{R}^3$, acting as translations on the space $M = \mathbb{R}^3$ and $\omega = \frac{1}{6}\epsilon_{abc}\,\mathrm{d}m^a \wedge \mathrm{d}m^b \wedge \mathrm{d}m^c$, the volume form, where ϵ_{abc} is the standard totally anti-symmetric Levi-Civita tensor. We shall use $a, b, c = 1, 2, 3$ as the indices of both the Lie algebra $\mathfrak{g} = \mathbb{R}^3$ and the coordinates $\{(m^1, m^2, m^3)\}$ on M. Since

$$i_a \omega = \tfrac{1}{2}\epsilon_{abc}\, \mathrm{d}m^b \wedge \mathrm{d}m^c = \tfrac{1}{2}\epsilon_{abc}\, \mathrm{d}\,(m^b\, \mathrm{d}m^c), \tag{4.1}$$

we can choose

$$\lambda_a = \tfrac{1}{2}\epsilon_{abc}\, m^b\, \mathrm{d}m^c. \tag{4.2}$$

It is clear that the cocycle

$$\gamma_{ab} = L_a \lambda_b = -\tfrac{1}{2}\epsilon_{abc}\, \mathrm{d}m^c \tag{4.3}$$

in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g} \otimes_{\mathbb{R}} \mathfrak{g}, Z^1(M))$ can not be a coboundary, for if this were the case, there would exist closed (hence exact) one-forms $\beta_a = \mathrm{d}f_a$ such that

$$\gamma_{ab} = L_a \beta_b = \mathrm{d}\, L_a f_b. \tag{4.4}$$

Comparing (4.3) and (4.4), we get

$$L_a f_b = -\tfrac{1}{2}\epsilon_{abc}\, m^c + g_{ab} \tag{4.5}$$

for a set of constants g_{ab} and hence

$$L_c L_a f_b = -\tfrac{1}{2}\epsilon_{abc}, \tag{4.6}$$

a contradiction since two partial derivatives commute (not anti-commute). Indeed, the relevant cohomology $H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, Z^1(M)))$ is not zero. To compute this group, notice first that as a $\mathfrak{g}$-module, $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, Z^1(M)) = Z^1(M) \oplus Z^1(M) \oplus Z^1(M)$. So what we shall compute is the cohomology group $H^1(\mathfrak{g}, Z^1(M))$. The space of the cocycles in $\mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, Z^1(M))$ is

$$\ker\delta_1 = \{(\beta_a) \mid \beta_a = \mathrm{d}f_a \in Z^1(M) = B^1(M), L_a\beta_b = L_b\beta_a\}, \tag{4.7}$$

whereas that of the coboundaries is

$$\mathrm{im}\delta_0 = \{(\beta_a) \mid \beta_a = L_a\beta = \mathrm{d}L_a f, \beta = \mathrm{d}f \in Z^1(M) = B^1(M)\}. \tag{4.8}$$

(4.7) implies $\mathrm{d}(L_a f_b - L_b f_a) = 0$ and hence

$$L_a f_b - L_b f_a = g_{ab}, \tag{4.9}$$

where g_{ab} are constants, among them three are independent. If they all vanish, then there is a function f such that $f_a = L_a f$ and hence

$$\beta_a = \mathrm{d}L_a f = L_a \mathrm{d}f \tag{4.10}$$

is a coboundary. Thus $H^1(\mathfrak{g}, Z^1(M)) \cong \mathbb{R}^3$ and

$$\mathrm{Ext}^1_{U(\mathfrak{g})}(\mathfrak{g}, Z^1(M)) \cong H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, Z^1(M))) \cong \mathbb{R}^9. \tag{4.11}$$

Finally, we wish to point out the consistency of our main result with the point of view of spectral sequences. Equivariant extension fits into a more general problem: given a fibration $F \overset{i}{\hookrightarrow} E \overset{\pi}{\to} B$, whether a cohomology class on F lies in the image of the pull back i^* from $H^*(E)$. In our case, $B = BG$, $F = M$, $E = EG \times_G M$.

Recall that every fibration determines a filtered cochain complex of the total space. For instance, if B is of a CW complex, we can use the natural filtration of the total space E according to the skeleta of B. A more relevant approach is the de Rham version [4]. Suppose E, F, B are smooth manifolds and $\{b\}$, $\{f\}$, local coordinates on B, F respectively. There is a natural filtration

$$0 = F^{n+1}\Omega^n(E) \subset F^n\Omega^n(E) \subset \cdots \subset F^1\Omega^n(E) \subset F^0\Omega^n(E) = \Omega^n(E) \tag{4.12}$$

of the de Rham complex $\Omega^*(E)$ where

$$F^p\Omega^n(E) = \Big\{\varphi = \sum_{|I|+|J|=n, |I|\geq p} \varphi_{IJ}(b,f)\,\mathrm{d}b^I \wedge \mathrm{d}f^J\Big\}. \tag{4.13}$$

Under a mild assumption that $\pi_1(B)$ acts trivially on $H^*(F)$ (which is obviously true in our case since $\pi_1(BG)$ is trivial), we can construct a spectral sequence such that $E_0^{pq} = \Omega^p(B, \Omega^q(F))$, the space of p-forms on B taking values in the space of q-forms on F, and $E_1^{pq} = \Omega^p(B, H^q(F))$. As usual, $E_2^{pq} = H^p(B, H^q(F))$ and the sequence converges to $H^*(E)$. The composite map in

$$H^q(E) \twoheadrightarrow E_\infty^{0q} = E_{q+1}^{0q} \subset E_q^{0q} \subset \cdots \subset E_2^{0q} = H^q(F) \tag{4.14}$$

is precisely the pull back $i^*: H^q(F) \to H^q(E)$ [8, ch. 3]. When the fibration is a non-trivial, the inclusions in (4.14) are proper, hence the Betti numbers of E are smaller than the corresponding ones of $B \times F$.

The cohomological obstructions in section 3 can be interpreted as the a series of conditions for the class $[\omega] \in H^q(F) = E_2^{0q}$ to lie in E_∞^{0q}, whose elements extend to the total space. In fact, a filtration of the space $\widetilde{\Omega} = (\Omega(M) \otimes S(\mathfrak{g}^*))^G$ of equivariant form is given by

$$F^p\widetilde{\Omega}^n = \Big\{ \sum_{r=[(p+1)/2]}^{[n/2]} \omega_{a_1\cdots a_r}^{(n-2r)} \phi^{a_1} \cdots \phi^{a_r} \mid \omega^{(n-2r)} \in \mathrm{Hom}_{U(\mathfrak{g})}(S^r\mathfrak{g}, \Omega^{n-2r}(M))\Big\}. \tag{4.15}$$

According to the standard algebraic construction [8, ch. 2],

$$E_k^{0q} = \frac{\widetilde{\Omega}^q \cap \widetilde{\mathrm{d}}^{-1}(F^{2[(k+1)/2]}\widetilde{\Omega}^{q+1})}{F^2\widetilde{\Omega}^q \cap \widetilde{\mathrm{d}}^{-1}(F^{2[(k+1)/2]}\widetilde{\Omega}^{q+1}) + \widetilde{\mathrm{d}}\widetilde{\Omega}^{q-1}}. \tag{4.16}$$

Therefore (4.14) becomes

$$H^q(\widetilde{\Omega}, \widetilde{\mathrm{d}}) \twoheadrightarrow E_\infty^{0q} = E_{2[q/2]+2}^{0q} = E_{2[q/2]+1}^{0q} \subset \cdots \subset E_4^{0q} = E_3^{0q} \subset E_2^{0q} = H^q(M) \tag{4.17}$$

for q even and

$$H^q(\widetilde{\Omega}, \widetilde{\mathrm{d}}) \twoheadrightarrow E_\infty^{0q} = E_{2[q/2]+3}^{0q} \subset E_{2[q/2]+2}^{0q} = E_{2[q/2]+1}^{0q} \subset \cdots \subset E_4^{0q} = E_3^{0q} \subset E_2^{0q} = H^q(M) \tag{4.18}$$

for q odd. The statement $[\omega] \in E_{2r+2}^{0q}$ means that ω can be extended to an element in $\widetilde{\Omega}^q$ which is "partially" closed, i.e., whose image under $\widetilde{\mathrm{d}}$ is in $F^{2r+2}\widetilde{\Omega}^{q+1}$. Thus for $[\omega]$ already in E_{2r}^{0q}, (A_r) and (B_r) in theorem 1 are precisely the conditions that guarantee $[\omega] \in E_{2r+2}^{0q}$. For q odd, the extra term in (4.18) that does not appear in (4.17) is clearly a reflection of the last equality in (3.3).

I would like to thank Y.-Q. Chen, L. Jeffrey and D. A. Vogan for discussions.

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